This week

1. Directives concerning the MLP test
2. Section 3.1: tangents and the derivative at a point
3. Section 3.2: the derivative as a function
4. Section 3.4: velocity
The (angle of) inclination is the angle $\theta$ that $\ell$ makes with the horizontal axis.

- The angle is measured from the positive $x$-axis to $\ell$.
- Turning counterclockwise means $\theta > 0$.
- Turning clockwise means $\theta < 0$.

The slope of a line

- The slope of $\ell$ is $\tan \theta = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.
- This holds for every choice $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, as long as $P_1 \neq P_2$. 
Equation of a line through a point with given slope

1.3

Let \( \ell \) be the line through \( P = (x_0, y_0) \) with slope \( m \), then for every point \( (x, y) \neq P \) on \( \ell \) we have

\[
\begin{align*}
  m &= \frac{y - y_0}{x - x_0} \\
  y - y_0 &= m(x - x_0) \\
  y &= m(x - x_0) + y_0.
\end{align*}
\]

The **equation of the line through \( P \) and with slope \( m \)** is

\[
y = m(x - x_0) + y_0
\]

Equation of a line with given slope and \( y \)-intercept

1.4

Let \( \ell \) be the line through with slope \( m \) and with \( y \)-intercept \( b \), then \( \ell \) passes through \((0, b)\).

The equation of \( \ell \) is \( y = m(x - 0) + b \), simplified:

\[
y = mx + b
\]
The derivative of a function

We define the **derivative of** \( f \) **at** \( x_0 \) as

\[
f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.
\]

The number \( f'(x_0) \) can be interpreted as:

- the slope of the graph of \( y = f(x) \) at the point \( (x_0, f(x_0)) \);
- the slope of the tangent line to the graph of \( y = f(x) \) at the point \( (x_0, f(x_0)) \);
- the rate of change of \( f(x) \) at the point \( x_0 \).

**Example**

**Example**

*Calculate the derivative of* \( f(x) = x^2 \) *at 1 with the definition.*

<table>
<thead>
<tr>
<th>( h )</th>
<th>( 1 + h )</th>
<th>( f(1) )</th>
<th>( f(1+h) )</th>
<th>( f(1+h)-f(1) )</th>
<th>( \frac{f(1+h) - f(1)}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
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<tr>
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<td>1.5</td>
<td>1</td>
<td>2.25</td>
<td>1.25</td>
<td>2.5</td>
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<td>1.5625</td>
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<tr>
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<td>1</td>
<td>1.002001</td>
<td>0.002001</td>
<td>2.001</td>
</tr>
</tbody>
</table>

This suggests: when \( h \) approaches 0, then \( \frac{f(1+h) - f(1)}{h} \) approaches 2.
Example 2.3

Calculate the derivative of \( f(x) = x^2 \) at 1 with the definition.

\[
\frac{f(1 + h) - f(1)}{h}
\]

Example 2.4

- The tangent line has slope \( f'(1) = 2 \) and passes through \((1, f(1)) = (1, 1)\).
- Hence the tangent line is described by the equation 

\[ y = 2x - 1 \]
Example 2.5

**Example**

Calculate the derivative of \( f(x) = x^2 \) at \( a \) with the definition.

\[
\frac{f(a + h) - f(a)}{h} =
\]

Example 2.6

**Example**

Calculate the derivative of \( f(x) = \sqrt{x} \) at \( a \) with the definition.

\[
\frac{f(a + h) - f(a)}{h} =
\]
Example

Calculate the derivative of \( f(x) = \frac{1}{x} \) at \( a \neq 0 \) with the definition.

\[
\frac{f(a + h) - f(a)}{h} = \frac{1}{a + h} - \frac{1}{a}
\]

The derivative as a function

**Definition**

The derivative of the function \( f \) is the function \( f' \) whose value at \( x \) is

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

- The function \( f \) is **differentiable at** \( x \) if \( f'(x) \) exists.
- The process of calculating \( f' \) is called **differentiation**.
- Alternative notations for the derivative are
  
  \[
  \frac{df}{dx} \quad \text{and} \quad \frac{d}{dx} f(x).
  \]
Example 3.2

On slide 10: the derivative of \( f \) at \( a \) is \( f'(a) = 2a \).

Replace \( a \) by \( x \): the derivative of \( f \) is the function \( f'(x) = 2x \).

Example 3.3

On slide 11: \( f'(a) = \frac{1}{2\sqrt{a}} \).

Replace \( a \) by \( x \): \( f'(x) = \frac{1}{2\sqrt{x}} \) \( (x > 0) \)
Example: the derivative of $f(x) = 1/x$

On slide 12: the derivative of $f$ at $a$ is $f'(a) = -\frac{1}{a^2}$.

Replace $a$ by $x$: $f'(x) = -\frac{1}{x^2}$ \ ($x \neq 0$)

Powers of $x$

Theorem

For all real numbers $\alpha$ we have

$$\frac{d}{dx} (x^\alpha) = \alpha x^{\alpha-1}$$

Check:

- Let $\alpha = \frac{1}{2}$, then
  $$\frac{d}{dx} \left( x^{\frac{1}{2}} \right) =$$

- Let $\alpha = -1$, then
  $$\frac{d}{dx} \left( x^{-1} \right) =$$
Average velocity

Consider a moving object and assume that we know the traveled distance as a function of time $s(t)$.

- If the object moves from $s(t_A)$ to $s(t_B)$, the displacement is $s(t_B) - s(t_A)$.
- The average velocity over the interval $(t_A, t_B)$ is the displacement per elapsed time, and is equal to
  \[
  \frac{s(t_B) - s(t_A)}{t_B - t_A}.
  \]

Velocity

Consider a moving object and assume that we know the traveled distance as a function of time $s(t)$.

- The velocity at time $t_A$ is the limit of the average velocity over the interval $(t_A, t_B)$ where $t_B$ approaches $t_A$:
  \[
  v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A}.
  \]
Velocity

\[ v(t_A) = \lim_{{t_B \to t_A}} \frac{s(t_B) - s(t_A)}{t_B - t_A}. \]

Define \( h = t_B - t_A \), then

- \( t_B = t_A + h \) and
- “\( t_B \to t_A \)” is equivalent to “\( h \to 0 \).”

\[ v(t_A) = \lim_{{t_B \to t_A}} \frac{s(t_B) - s(t_A)}{t_B - t_A} = \lim_{{h \to 0}} \frac{s(t_A + h) - s(t_A)}{h} = s'(t_A). \]

**Velocity is the derivative of displacement**

Average acceleration

Consider a moving object and assume that we know the velocity as a function of time \( v(t) \).

\[ v(t_A) \quad v(t_B) \]
\[ t = t_A \quad t = t_B \]

- The **average acceleration over the interval** \((t_A, t_B)\) is the change in velocity per elapsed time, and is equal to

\[ \frac{v(t_B) - v(t_A)}{t_B - t_A} \]
**Acceleration**

The acceleration at time $t_A$ is the limit of the average acceleration over the interval $(t_A, t_B)$ where $t_B$ approaches $t_A$:

$$a(t_A) = \lim_{t_B \to t_A} \frac{v(t_B) - v(t_A)}{t_B - t_A} = v'(t_A).$$

**Acceleration is the derivative of velocity**

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**Higher order derivatives**

**Definition**

Let $n$ be a non-negative integer. The $n$-th derivative of $f$ is denoted as $f^{(n)}$ or $\frac{d^n f}{dx^n}$, and is defined as

$$f^{(n)}(x) = \begin{cases} f(x) & \text{if } n = 0, \\ f'(x) & \text{if } n = 1, \\ \frac{d}{dx} \left( f^{(n-1)}(x) \right) & \text{otherwise.} \end{cases}$$

- The second derivative is denoted as $f''$ and not as $f^{(2)}$.
- Acceleration is the second derivative of displacement: $a(t) = s''(t)$. 

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Which is better: falling or crashing?

4.7

50 km/h

10 m

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Capacitors

4.8

**Physical principles**

(1) *In a capacitor, the charge $Q$ on the plates is proportional to the voltage $V$ over the plates: hence $Q = CV$, where $C$ is the capacity.*

(2) *The current through a lead is the amount of charge per second flowing through the lead.*